Physics 208 Exam 3 **Name (printed)** Solution

On my honor as a Texas A&M University at Qatar student, I will neither give nor receive unauthorized help on this exam.

Name (signed) Solution !

You are graded on your work, with partial credit. (Be sure to include the proper units in each answer.)
The answer by itself is not enough, and you receive credit only for your work.
Please be clear and well-organized, so that we can easily follow each step of your work.
See the last page of the exam for the formula sheets.

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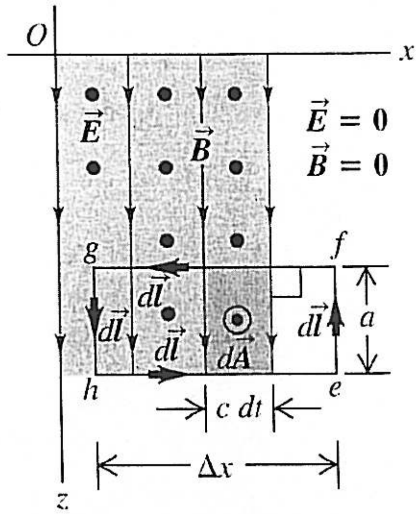
1. In doing this problem, do not assume any relationship between c and ε_0 and μ_0, since the point of this problem is to obtain this relationship. You may only assume Maxwell's equations.



(a) The figure represents the simplest version of an electromagnetic wave which is propagating to the right. Within the shaded region, the uniform electric field points out of the paper, and the uniform magnetic field points down within the paper, as shown. The fields to the right of the shaded region are equal to zero. Within the time dt, the wave front moves to the right by $c\,dt$.

(i) (3) Calculate the value of the magnetic field line integral around the loop $efgh$.

$\int \vec{B} \cdot d\vec{\ell} = Ba$ [since only gh contributes]

(ii) (3) Calculate the value of the extra electric flux added to the surface area within this loop during the time dt.

$d\Phi_E = E \cdot c\,dt \cdot a$

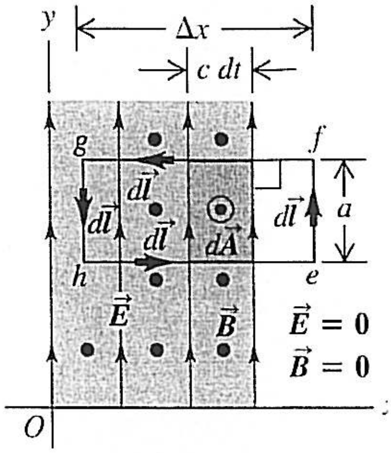
(iii) (3) Using Ampere's law (as generalized by Maxwell), relate the magnitudes of the magnetic field $\vec{B}$ and the electric field $\vec{E}$. I.e., get $B =$ constant $\times$ E while at the same time obtaining this constant in terms of fundamental constants like c, ε_0, and μ_0. **Your calculation must be based on your results of part (i) and part (ii) above.**

$$\int \vec{B} \cdot d\vec{\ell} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\Rightarrow \quad Ba = \mu_0 \epsilon_0 \cdot E \cdot c \cdot a$$

$$\Rightarrow \quad \boxed{B = \mu_0 \epsilon_0 c E}$$

(b) Now let us turn to the second figure, shown on this page. This figure again represents the simplest version of an electromagnetic wave which is propagating to the right. Now, within the shaded region, the uniform magnetic field points out of the paper, and the uniform electric field points up within the paper, as shown. The fields to the right of the shaded region are equal to zero. Within the time dt, the wave front moves to the right by $c\,dt$.



(i) (3) Calculate the value of the electric field line integral around the loop $efgh$.

$\oint \vec{E}\cdot d\vec{l} = -Ea$ [since $\vec{E}$ is upward]

(ii) (3) Calculate the value of the extra magnetic flux added to the surface area within this loop during the time dt.

$d\Phi_B = B\cdot c\,dt\cdot a$

(iii) (3) Using Faraday's law, relate the magnitudes of the electric field $\vec{E}$ and the magnetic field $\vec{B}$. I.e., get E = constant × B while at the same time obtaining this constant in terms of fundamental constants like c, ε_0, and μ_0. **Your calculation must be based on your results of part (i) and part (ii) above.**

$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\frac{d\Phi_B}{dt}$

$\Rightarrow -Ea = -B\cdot c\cdot a \Rightarrow \boxed{E = cB}$

(c) (2) Use the results of parts (a) and (b) to obtain the speed of electromagnetic waves in terms of the other constants.

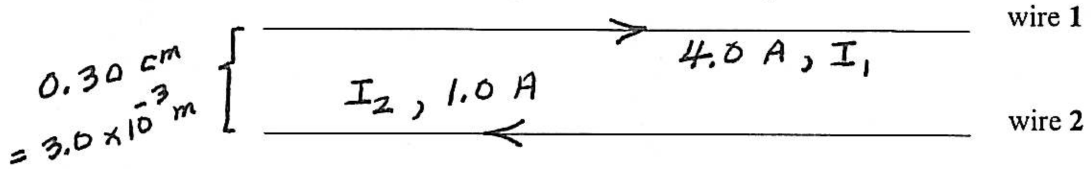
$E = cB = c\cdot\mu_0\epsilon_0 cE$

$\Rightarrow 1 = c^2\mu_0\epsilon_0$

$\Rightarrow \boxed{c = \frac{1}{\sqrt{\epsilon_0\mu_0}}}$

2. Two long straight parallel wires are separated by a distance of 0.30 cm. The current in wire **1** is 4.0 A, and it flows toward the right. The current in wire **2** is 1.0 A, and it flows toward the left.

(a) (2) Complete the picture below showing these wires, their separation, and their currents.



(b) (3) Calculate the magnitude of the magnetic field $\vec{B}_1$ due to wire **1** at the position of wire **2**.

Answer: magnitude of $\vec{B}_1$ = 2.7×10^{-4} T

$$\oint \vec{B}_1 \cdot d\vec{l} = \mu_0 I_1$$

$$\Rightarrow B_1 \cdot 2\pi r = \mu_0 I_1$$

$$\Rightarrow \boxed{B_1 = \frac{\mu_0 I_1}{2\pi r}} = \left(4\pi \times 10^{-7} \frac{Wb}{m \cdot A}\right) \frac{4.0 \text{ A}}{(2\pi)(3.0 \times 10^{-3} \text{ m})}$$

$$= \boxed{2.7 \times 10^{-4} \text{ T}} \qquad \left[\frac{Wb}{m^2} = T \right]$$

[other version : $B_1 = 2.0 \times 10^{-4}$ T]

(c) (2) What is the direction of $\vec{B}_1$?
(into paper, out of paper, to right, to left, up toward top of paper, down toward bottom of paper)

Answer: direction of $\vec{B}_1$ is into paper

(d) (3) Calculate the magnitude of the force per unit length on wire **2** (due to $\vec{B}_1$).

Answer: magnitude of force per unit length on wire **2** = $2.7 \times 10^{-4} \frac{N}{m}$

$$F = I_2 L B_1$$

$$\Rightarrow \boxed{\frac{F}{L} =} I_2 B_1 = (1.0 \text{ A})(2.7 \times 10^{-4} \text{ T})$$

$$= \boxed{2.7 \times 10^{-4} \frac{N}{m}}$$

$$\left[\vec{F} = q\vec{v} \times \vec{B} \Rightarrow N = (C)\left(\frac{m}{s}\right)(T) \right.$$

$$\left. \Rightarrow A \cdot T = \frac{N}{m} \right]$$

[other version: $\frac{F}{L} = 1.2 \times 10^{-3} \frac{N}{m}$]

(e) (3) Calculate the magnitude of the magnetic field $\bar{B}_2$ due to wire **2** at the position of wire **1**.

Answer: magnitude of $\bar{B}_2$ = 6.7×10^{-5} T

As in part (b),

$$\boxed{B_2 =} \frac{\mu_0 I_2}{2\pi r} = \left(4\pi \times 10^{-7} \frac{Wb}{m \cdot A}\right) \frac{1.6\ A}{(2\pi)(3.0 \times 10^{-3}\ m)}$$

$$= \boxed{6.7 \times 10^{-5}\ T}$$

[other version: $B_2 = 6.0 \times 10^{-4}$ T]

(f) (2) What is the direction of $\bar{B}_2$?
(into paper, out of paper, to right, to left, up toward top of paper, down toward bottom of paper)

Answer: direction of $\bar{B}_2$ is into paper

(g) (3) Calculate the magnitude of the force per unit length on wire **1** (due to $\bar{B}_2$).

Answer: magnitude of force per unit length on wire **1** = $2.7 \times 10^{-4}\ \frac{N}{m}$

Here $F = I_1 L B_2$

$$\Rightarrow \boxed{\frac{F}{L} =} I_1 B_2 = (4.0\ A)(6.7 \times 10^{-5}\ T)$$

$$= \boxed{2.7 \times 10^{-4}\ \frac{N}{m}}$$ in this part

consistent with part (d)

[Newton's 3rd law, action force = reaction force]

[other version: $\frac{F}{L} = 1.2 \times 10^{-3}\ \frac{N}{m}$ in this part]

(h) (2) Do the wires repel or attract each other? Explain.

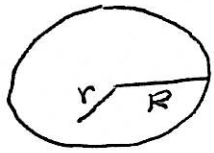
right-hand rule ⇒ upward force on wire 1 and downward force on wire 2

so they repel each other

3. A long, straight, solid cylinder, with radius R, is oriented with its axis in the z direction. It carries a current I whose current density is $\vec{J}$. The current density, although symmetric about the cylinder axis, is not constant, but varies according to

$$J = \alpha r$$

where α is a constant.



(a) (5) Calculate the value of α in terms of I and R.

$$(\text{area in } dr \text{ at } r) = 2\pi r\,dr$$

$$\Rightarrow (\text{current in } dr \text{ at } r) = J \cdot 2\pi r\,dr = \alpha r \cdot 2\pi r\,dr$$

$$\text{Then} \quad I = \int_0^R \alpha r \cdot 2\pi r\,dr = 2\pi\alpha \left[\frac{r^3}{3}\right]_0^R = 2\pi\alpha\frac{R^3}{3}$$

$$\Rightarrow \boxed{\alpha = } \frac{I}{2\pi R^3/3} = \boxed{\frac{3}{2\pi}\frac{I}{R^3}} \qquad \left[\frac{A}{m^3} \text{ is correct units}\right]$$

(b) (5) Using Ampere's law, calculate the magnitude $B(r)$ of the magnetic field in the region $r \geq R$. Express your answer in terms of I.

$$\underbrace{\oint \vec{B}\cdot d\vec{l}}_{=2\pi r \cdot B} = \mu_0 I_{enclosed} = \mu_0 I$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi r}}$$

(c) (10) Again using Ampere's law, calculate $B(r)$ in the region $r \leq R$. And again express your answer in terms of I.

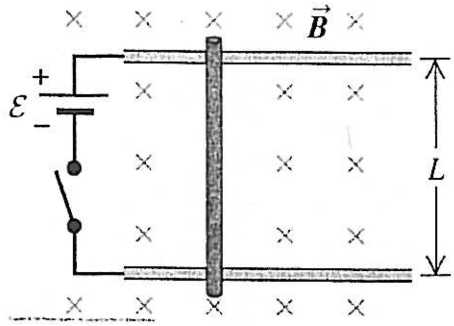
$$\underbrace{\oint \vec{B}\cdot d\vec{l}}_{2\pi r \cdot B(r)} = \mu_0 \int_0^r \alpha r' \cdot 2\pi r'\,dr' \qquad \text{as in part (a)}$$

$$= \mu_0 \cdot \cancel{2\pi} \cdot \frac{\cancel{3}}{\cancel{2\pi}}\frac{I}{R^3}\left[\frac{r'^3}{\cancel{3}}\right]_0^r$$

$$= \mu_0 I \frac{r^3}{R^3}$$

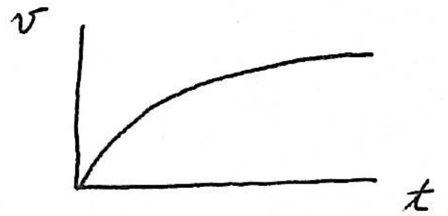
$$\Rightarrow \boxed{B(r) =} \frac{\mu_0 I}{2\pi r}\frac{r^3}{R^3} = \boxed{\frac{\mu_0 I}{2\pi R^3} r^2}$$

4. A bar of length $L = 0.5$ m is free to slide without friction on horizontal rails, as shown in the figure. There is a uniform magnetic field $B = 1.0\,\mathrm{T}$ directed into the plane of the figure. At one end of the rails there is a battery with emf $\varepsilon = 3$ V and a switch. The bar has mass $m = 0.4$ kg and resistance $R = 5.0\,\Omega$, and all other resistance in the circuit can be ignored. The switch is closed at time $t = 0$.



(a) (5) Draw a qualitative sketch of the speed of the bar as a function of time.

The bar will initially accelerate, but will finally reach a velocity for which the induced emf will cancel the battery's emf: No current ⇒ no force.



(b) (5) Just after the switch is closed, what is the acceleration of the bar?

$$I = \frac{\varepsilon}{R} = \frac{3\ \mathrm{V}}{5.0\ \Omega} = 0.6\ \mathrm{A}$$

$$F = ILB = (0.6\ \mathrm{A})(0.5\ \mathrm{m})(1.0\ \mathrm{T})$$

$$= 0.3\ \mathrm{N}$$

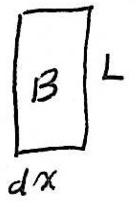
$$\boxed{a =} \frac{F}{m} = \frac{0.3\ \mathrm{N}}{0.4\ \mathrm{kg}} = \boxed{0.75\ \frac{\mathrm{m}}{\mathrm{s}^2}}$$

[other version: $I = 2\ \mathrm{A}$, $F = 2\ \mathrm{N}$, $a = 10\ \mathrm{m/s^2}$]

(c) (5) What is the acceleration of the bar when its speed is 4.0 m/s?

$$\mathcal{E}_{induced} = -\frac{d\Phi_B}{dt}$$

$$d\Phi_B = B \cdot L \cdot dx \Rightarrow \frac{d\Phi_B}{dt} = BLv$$



$$\therefore \mathcal{E}_{induced} = -BLv$$

$$= -(1.0\ T)(0.5\ m)(4\ \tfrac{m}{s})$$

$$\frac{dx}{dt} = v$$

$$= -2\ V$$

Now $I = \dfrac{3V - 2V}{5.0\ \Omega} = 0.2\ A \Rightarrow F = (0.2A)(0.5\ m)(1.0T) = 0.1\ N$

$$\Rightarrow \boxed{a =} \frac{0.1\ N}{0.4\ kg} = \boxed{0.25\ \tfrac{m}{s^2}}$$

[other version: $\mathcal{E}_{induced} = -4\ V$, $I = 0.67\ A$,
$F = 0.67\ N$, $a = 3.3\ m/s^2$]

(d) (5) What is the terminal speed of the bar?

current = 0 when $\mathcal{E}_{induced} + \mathcal{E} = 0$, or

$$BLv = \mathcal{E} \Rightarrow \boxed{v =} \frac{\mathcal{E}}{BL}$$

$$= \frac{3V}{(1.0\ T)(0.5\ m)}$$

$$= \boxed{6\ \tfrac{m}{s}}$$

[As in problem 2, $A \cdot T = \frac{N}{m}$ and $C \cdot V = J$, so

$$\frac{V}{T \cdot m} = \frac{J/C}{\frac{N/m}{A} \cdot m} = \frac{J/C}{\frac{N}{C/s}} = \frac{J}{N} \cdot \frac{C}{s} \cdot \frac{1}{C} = \frac{m}{s}.]$$

[other version: $v = 6\ m/s$]

5. As engineers in the real world, you and your colleagues invent a device that will be useful for the electronics that govern an oil refinery. It is called a Qator, and it has the property that the potential change V_Q across a Qator is

$$V_Q = -\overline{Q}\frac{d^3 i(t)}{dt^3} = -\overline{Q}\frac{d^4 q(t)}{dt^4} .$$

Here $\overline{Q}$ is a constant, which is not related to the charge $q(t)$. The Qator is connected in series with a capacitor having capacitance C, and a charge $q(0)$ is placed on the capacitor at time $t = 0$. (There are no other elements in this QC circuit.)

(a) (i) (5) Write down the differential equation for the charge $q(t)$, in terms of $\overline{Q}$ and C.

$$-\overline{Q}\frac{d^4 q(t)}{dt^4} + \frac{q(t)}{C} = 0$$

[Kirchhoff loop rule, $V_C = \frac{q}{C}$]

(ii) (5) Calculate the period of oscillation T of a QC circuit in terms of $\overline{Q}$ and C.

There is more than one way.

One way: try $q(t) = A\cos(\omega t + \phi)$

Then $\frac{d^4 q}{dt^4} = A\omega \frac{d^3}{dt^3}(-\sin(\omega t+\phi))$

$= A\omega\cdot\omega \frac{d^2}{dt^2}(-\cos(\omega t+\phi))$

$= A\omega^2\cdot\omega \frac{d}{dt}(+\sin(\omega t+\phi))$

$= A\omega^3\cdot\omega \cos(\omega t+\phi)$

so $-\overline{Q}\cdot A\omega^4 \cos(\omega t+\phi) + \frac{1}{C}\cdot A\cos(\omega t+\phi) = 0$

$\Rightarrow -\overline{Q}\omega^4 + \frac{1}{C} = 0$

$\Rightarrow \boxed{\omega = \frac{1}{(\overline{Q}C)^{1/4}}}$

and $\boxed{T =} \frac{2\pi}{\omega} = \boxed{2\pi(\overline{Q}C)^{1/4}}$

[If we take $\phi = 0$, $A = q(0)$.]

(b) Now the capacitor is replaced by a resistor with resistance R, so that we have a QR circuit. I.e., the only circuit elements are the Qator and the resistor.

(i) (5) Write down the differential equation for the charge $q(t)$ in this case, in terms of $\overline{Q}$ and R.

Ohm's law : $V_R = i(t) R = \frac{dq}{dt} R$

We can use either $i(t)$ or $q(t)$, If we use $q(t)$,

$$-\overline{Q} \frac{d^4 q(t)}{dt^4} - \frac{dq(t)}{dt} R = 0$$

(ii) (5) Show that the charge decays with time according to

$$q(t) = q(0)\, e^{-t/\tau}$$

while at the same time calculating the decay constant τ in terms of $\overline{Q}$ and R.

Using this trial solution, we have

$$\frac{d^4 q}{dt^4} = q(0) \left(-\frac{1}{\tau}\right)^4 e^{-t/\tau}$$

since $\frac{d}{dt}$ brings down a $-\frac{1}{\tau}$

Then $-\overline{Q} \frac{1}{\tau^4} q(t) - \left(-\frac{1}{\tau}\right) q(t) R = 0$

$$\Rightarrow \quad -\overline{Q} \frac{1}{\tau^3} + R = 0$$

$$\Rightarrow \quad \boxed{\tau = \frac{1}{(\overline{Q} R)^{1/3}}}$$

6. (5 points extra credit) Describe 5 **distinct** applications of Faraday's law. We mentioned, e.g.,

(i) generators (or equivalently, alternators)

(ii) credit card and ATM card readers

(iii) metal detectors

(iv) electric guitars

(v) transformers

(vi) magnetic computer memories

7. (5 points extra credit) Suppose that magnetic monopoles were discovered, so that we would have a magnetic charge Q_M as well as an electric charge Q_E, and a magnetic current i_M as well as an electric current i_E. Write down Maxwell's equations as they would then appear, in terms of Q_M, i_M, etc. As always, please be clear.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_E}{\epsilon_0}$$

$$\oint \vec{B} \cdot d\vec{A} = \mu_0 Q_M$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{\partial \Phi_M}{\partial t} - \mu_0 i_M$$

$$\oint \vec{B} \cdot d\vec{l} = \epsilon_0 \mu_0 \frac{\partial \Phi_E}{\partial t} + \mu_0 i_E$$

[exact constants and signs are not counted as important here]